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Fuzzy Finite Element Approach for the Analysis of Imprecisely Defined Systems

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Many engineering systems are too complex to be defined in precise mathematical terms. They often contain information and features that are vague, imprecise, qualitative, linguistic, or incomplete. The traditional deterministic and probabilistic techniques are not adequate to analyze such systems. This paper aims at developing a fuzzy finite element approach for the analysis of imprecisely defined systems. The development of the methodology starts from the basic concepts of fuzzy numbers of fuzzy arithmetic and implements suitably defined fuzzy calculus concepts such as differentiation and integration for the derivation, manipulation, and solution of the finite element equations. Simple stress analysis problems involving vaguely defined geometry, material properties, external loads, and boundary conditions are solved to establish and to illustrate the new procedure. The approach developed is applicable to systems that are described in linguistic terms as well as those that are described by incomplete information. If complete data are known, the method handles the information similar to that of a probabilistic approach. The present approach represents a unique methodology that enables us to handle certain types of imprecisely known data more realistically compared with the existing procedures.

I. Introduction

DEPENDING on the nature and extent of uncertainty present in an engineering system, three approaches can be used for its analysis as indicated in the uncertainty triangle of Fig. 1. If the system parameters are treated as random variables with known probability distributions, the performance or output of the system can be determined using the theory of probability and random processes. If only a fragmentary information on the uncertainty quantity is available, it is possible to establish an upper bound on the maximum response of the system using the antioptimization approach. On the other hand, if the system parameters are described in linguistic or imprecise terms, fuzzy theory can be used to predict the response.

Many real-life engineering problems are too complex and ill defined to be modeled by conventional deterministic procedures. These problems contain fuzzy information, that is, information that is vague, imprecise, qualitative, linguistic, or incomplete. The subjective judgment of the analyst affects the assumptions on which an analysis is based as well as the interpretation and use of the results of analysis. The fuzzy or imprecise information may be present in the geometry, material properties, applied loads, or boundary conditions of a system. In the traditional (deterministic) finite element approach, all of the parameters of the system are assumed to be precisely known. The stochastic or probabilistic finite element method was developed to account for uncertainties in the geometry or material properties of the structure, as well as the applied loads. These uncertainties are spatially distributed over the region of the structure and are modeled as stochastic or random fields. However, certain uncertainties, especially those involving descriptive and linguistic variables as well as those based on very scarce information, cannot be handled satisfactorily in the stochastic finite element approach. For example, the statements such as "the load acting on the beam is around 1000 1b," "the Young's modulus of the material is substantially large than 28,000 ksi," "the fiber content of the composite plate is very low," or "the boundary conditions are very close to (but not same as) fixed supports," are vague and cannot be handled by the existing approaches. In machine and structural design, the failure of a machine or structure is established based on a failure theory such as the von Mises criterion and the limiting (or permissible) stress is related to the yield stress of the material. In many cases, the yield stress may not be known precisely. For example, if the permissible stress is taken as $\sigma_{max} = 20{,}000$ psi, it implies that an induced (equivalent) stress of $\sigma = 20{,}000$ psi will not cause any failure, whereas $\sigma = 20{,}001$ psi will cause failure. However, there is no substantive difference between 20,000 psi and 20,001 psi. The conventional design procedures do not permit an induced stress of 20,001 psi, but a designer, more realistically, may permit it with slightly less than 100% satisfaction. In fact, the designer may permit even a larger induced stress with a lower satisfaction level. Beyond a certain value of the induced stress, it may be completely unsatisfactory to the designer. Thus it appears that it is more reasonable to assume a transition stage from absolute permission to absolute impermission when the allowable interval of the induced stress is determined. This implies that the ordinary subset is to be replaced by a fuzzy subset along the real axis for the permissible stress. In some applications, the boundary conditions may not be known precisely. For example, the exact support conditions of the gudgeon pin in a piston, the clevis pin in a knuckle joint, or the arbor in a milling machine are not known. The bolts between a machine and its foundation may not have been tightened properly. The actual boundary conditions in all these cases lie some where between simple supports and fixed ends. Similar vagueness is encountered in handling contact stress problems. This paper presents a new methodology, based on fuzzy set theory and finite element method, for analyzing the various types of fuzzy systems indicated earlier.

II. Brief Literature Review

A. Fuzzy Set Theory and Applications

Fuzzy set theory was initiated by Zadeh in 1965.[1] Since then, for some 10 years, the mathematics of the subject was developed by various researchers, but few applications resulted. In fact, in the early days, because of the subjective aspects involved, fuzzy set methods have often been dismissed prematurely by investigators who are unfamiliar with the area. However, recent applications of the subject to various scientific areas, such as artificial intelligence, image processing, speech recognition, biological and medical sciences, decision theory, economics, geography, sociology, psychology, linguistics, and semiotics, have indicated that fuzzy set theory may not be a theory in search of applications but indeed a useful tool for the quantification of impreciseness and vagueness present in many real-life problems. Most engineering applications of fuzzy set theory have been related to controls, decision making, and optimization.

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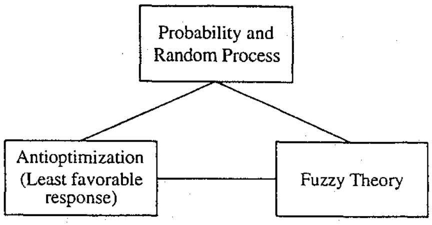

Fig. 1 Uncertainty triangle.

A large number of papers have been written on fuzzy set theory as well as its engineering applications. Brown and Yao[2] and Brown[3] discussed the application of fuzzy set theory to structural design. The fuzzy optimum design of structures was studied by Yuan and Quan.[4] The application of fuzzy methodologies to the multiobjective optimization of mechanical and structural systems was presented by Rao.[5,6] The use of nonlinear membership functions in the optimization of engineering systems was proposed and demonstrated by Dhingra et al.[7] The fuzzy random vibration with fuzzy parameters, with application to aseismic structures, was considered by Wang and Ou.[8] It can be seen from a review of the available literature that fuzzy set theory has been applied for decision-making problems but not for the analysis of engineering systems. This work extends the fuzzy set theory for the finite element analysis of engineering systems containing vague information.

B. Stochastic Finite Element Method

Although there is no suitable technique available for the analysis of all types of imprecision, the stochastic finite element method can be used to handle uncertain parameters that are described by probability distributions. The stochastic finite element method was developed in the 1980s to account for uncertainties in the system parameters, geometry, and external actions. The uncertain variables were modeled as random variables/ random fields with known characteristics. In 1980, a generic stochastic finite element method was proposed by Contreras[9] for modeling and analyzing structures in a probabilistic framework. The transient structural loads, idealized as stochastic processes, were incorporated into finite element dynamic models with uncertain parameters. Handa and Anderson[10] presented a finite element technique that gives estimates of the mean values, standard deviations, and correlation coefficients of structural displacements and stresses by taking into account variations in applied loads, dimensions, and material properties. Nakagiri et al.[11] presented a method for the uncertain eigenvalue analysis of fiber-reinforced plastic (FRP) plates. By treating the fiber orientations and thicknesses of plies as random variables, the coefficients of variation of the eigenfrequency were evaluated. Vanmarcke and Grigoriu[12] developed a method of stochastic finite element analysis for solving a variety of engineering mechanics problems in which physical properties exhibit one-dimensional spatial random variation.

III. Fuzzy Finite Element Method

In traditional finite element methods, the development of the element matrices and vectors requires the evaluation of integrals over the domain or region of the element. The resulting assembled finite element equations are to be solved using a suitable technique such as Gaussian elimination or one of its variants. In the case of a fuzzy system, the integrals of fuzzy quantities are to be evaluated over fuzzy domains. Similarly the methods of solving a system of equations are to be redefined to handle fuzzy information. All of these theoretical and computational aspects can be developed starting from the basic definitions of a fuzzy quantity and fuzzy arithmetic operators.

A. Construction of Membership Functions

The key elements of a vague, imprecise, or linguistic statement are not numbers but labels of fuzzy sets, that is, classes of objects in which the transition from membership (degree of belongingness) to nonmembership is gradual rather than abrupt as shown in Fig. 2b. In fact, the existence of fuzziness in linguistic statements (reflecting the human thought process) suggests that much of the logic behind human reasoning is not the traditional two-valued or even multivalued logic but a logic with fuzzy truths, fuzzy connectives, and fuzzy rules of inference. Fuzzy logic is an extension of set theoretic multivalued logic[13] in which the truth values are linguistic variables. As an example, the linguistic variable "fiber content" in an FRP composite beam may take the values of "low," "not low," "high," "very high," "high but not very high," etc., as its fuzzy subsets. There is a syntactic rule that generates the possible values of a linguistic variable. A semantic rule defines the meaning of each fuzzy variable over a universe of discourse. A fuzzy subset does not have a crisp boundary. It can be represented by a membership function, $\mu_A(x)$, which denotes the grade of membership of element x in the fuzzy subset A. If $\mu_A(x) = 1$ for some value of x, then this value is definitely a member of the fuzzy subset A. Similarly, $\mu_A(x) = 0$ implies that the particular x is definitely outside the fuzzy subset A. Any value of x in the range means that the membership of x in A is vaguely defined. Thus the membership function indicates the degree of possibility that a particular item is a member of the fuzzy subset A. The application of fuzzy methodologies requires a knowledge of the membership functions of the fuzzy quantities. Linear membership functions are commonly used in fuzzy set applications, mainly for reasons of computational simplicity. However, the realistic modeling of most practical situations necessitates the use of nonlinear membership functions. Several different shapes with positive (convex), negative (concave), or zero (linear) values of the coefficient of membership satiation can be used for modeling different types of imprecision. The membership function of a fuzzy set can be based on statistical data. It is to be noted that not all fuzzy quantities have statistical basis for defining their membership functions. For example, tall buildings, large objects, etc., have identifying features with

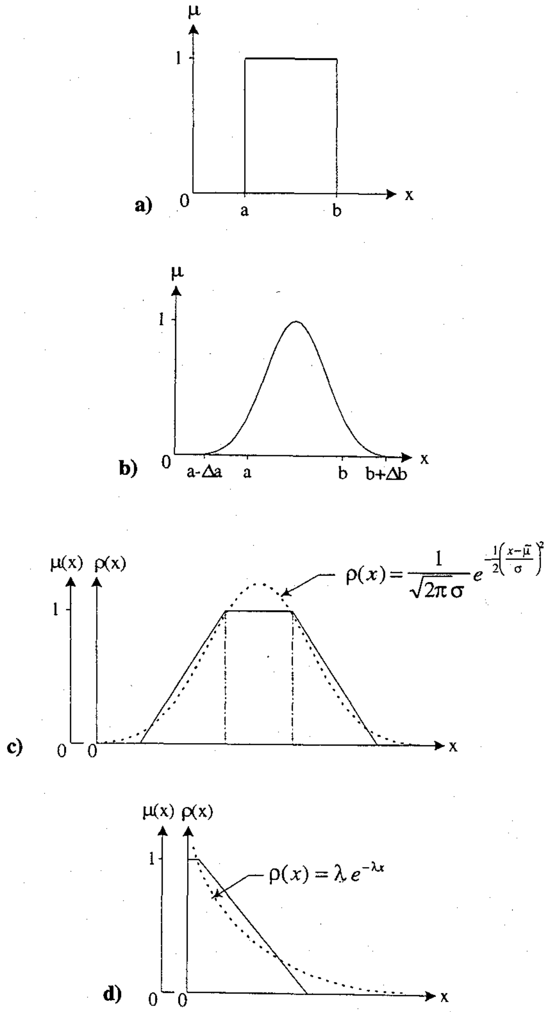

Fig. 2 Crisp and fuzzy sets: a) crisp set "high", b) fuzzy set "high", and c) Gaussian density function, and d) exponential density function.

infinite range of probable values. If statistical data are known, the membership function can be determined as

$$\mu(x) = \lambda p(x) \tag{1}$$

$$\lambda = \frac{1}{\max[p(x)]} \tag{2}$$

and $p(x)$ is the probability mass/density function or its estimate derived from the histogram of the feature (X) used for defining the fuzzy set. Equation (2) satisfies the possibility–probability consistency principle that can be stated as follows[14]: "The degree of possibility of an element is greater than or equal to its degree of probability." If the membership function is used as a grade for possibility, the consistency principle can be restated as

$$\max_{x\in D}\left\{\frac{\mu(x)}{\max \mu(x)}\right\} \geq \int_D p(x)\,\mathrm{d}x \tag{3}$$

for any set D on the real line. In some cases, the probability density function can be used to describe the membership function by preserving the overall shape of the density function. This is indicated for the Gaussian density function in Fig. 2c and the exponential density function in Fig. 2d.

B. Fuzzy Operations for Finite Element Analysis

Fuzzy Arithmetic

Fuzzy arithmetic is commonly presented as combinations of max-min operations performed on the membership values of the elements of fuzzy numbers. The addition of two fuzzy numbers A and B can be written as

$$\mu_{A(+)B}(Z) = \bigvee_{z=x+y} [\mu_A(x) \wedge \mu_B(y)] \qquad \forall x, y, z \in R \tag{4}$$

For example, if the fuzzy numbers A and B are defined as

$$A = 0/0 + 0.1/1 + 0.3/2 + 0.8/3 + 1/4 + 0.7/5 + 0.3/6 + 0/7$$

$$B = 0/0 + 0.3/1 + 0.6/2 + 1/3 + 0.7/4 + 0.2/5 + 0.1/6 + 0/7$$

then the sum $C = A + B$ is given by

$$C = 0/0 + 0/1 + 0.1/2 + 0.3/3 + 0.3/4 + 0.6/5 + 0.8/6 + 1/7 + 0.7/8 + 0.7/9 + 0.3/10 + 0.2/11 + 0.1/12 + 0/13 + 0/14$$

The operations of subtraction, multiplication, and division with fuzzy numbers can also be defined using the max-min convention. In general, for discrete fuzzy sets, the number of elements in the resulting fuzzy number C will vary depending on the nature of A and B. In this convention, the number of elements in the resulting fuzzy sets could become quite large. Thus, in numerical computations, it is convenient to express fuzzy numbers as sets of upper and lower bounds of a finite number of α-cut subsets. If F is a fuzzy number with a membership function μ_f, then an α-cut is defined as the set

$$F_a = \{x \in X, \mu_F(x) \geq \alpha\} \tag{5}$$

For the ith α-cut, the upper and lower bounds (f^L, f^R) are given by

$$f^L = \min\{f : f \in F_a\} \tag{6}$$

$$f^R = \max\{f : f \in F_a\} \tag{7}$$

If a set of n α-levels is established for all fuzzy quantities in a given problem, each of these quantities may be represented by a 2 by n array as follows:

$$F = \left\{(f^L, f^R)_1 (f^L, f^R)_2, \ldots, (f^L, f^R)_n\right\} \tag{8}$$

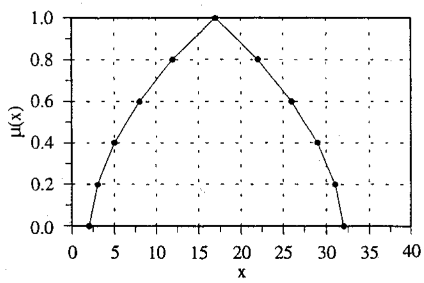

Fig. 3 Typical fuzzy number, A.

For examples, it may be desired to evaluate a function containing fuzzy numbers for the α-levels

$$\alpha = \{0.0, 0.2, 0.4, 0.6, 0.8, 1.0\}$$

For this case, the fuzzy number A shown in Fig. 3 can be written as

$$A = \{(2, 32), (3, 31), (5, 29), (8, 26), (12, 22), (17, 17)\}$$

With the fuzzy quantities expressed in this interval form, fuzzy arithmetic operations can be carried out using interval operations at each of the n α-levels independently. The addition of two fuzzy numbers A and B with

$$A = \left\{(a^L, a^R)_i\right\} \qquad \text{and} \qquad B = \left\{(b^L, b^R)_i\right\} \tag{9}$$

can then be expressed as

$$A(+)B = \left\{(a^L + b^L, a^R + b^R)_i\right\}; \qquad i = 1, 2, \ldots, n \tag{10}$$

The addition operation of fuzzy numbers is both commutative and associative. The subtraction of the fuzzy numbers A and B given by Eq. (9) can be expressed as

$$A(-)B = \left\{(a^L - b^R, a^R - b^L)_i\right\}; \qquad i = 1, 2, \ldots, n \tag{11}$$

The subtraction of fuzzy numbers is neither associative nor commutative. The multiplication of fuzzy numbers A and B can be expressed as

$$A(\times)B = \{\min[a^L \cdot b^L, a^L \cdot b^R, a^R \cdot b^L, a^R \cdot b^R] \;\; \max[a^L \cdot b^L, a^L \cdot b^R, a^R \cdot b^L, a^R \cdot b^R]\} \qquad i = 1, 2, \ldots, n \tag{12}$$

The multiplication of fuzzy numbers is both commutative and associative but is not distributive in $\Re$. The division of the fuzzy numbers A and B is defined as

$$A(\div)B = A(\times)B^{-1}; \qquad i = 1, 2, \ldots, n \tag{13}$$

The division of fuzzy numbers is neither associative nor commutative. The interval and max-min operations can be proved to be mathematically equivalent.[15]

Manipulation of Linguistic Variables

The fuzzy set theory can be used to describe linguistic variables also. For example, labels such as strong, weak, flexible, tall, and old; hedges such as very, quite, fairly, and extremely; negation (not); and connectives (and, but, or) can be assembled into relatively complex statements, and their fuzzy representations can be compounded with the operations indicated earlier. The fuzzy representations of typical linguistic statements are shown in Table 1.

Fuzzy Integration

If $f(x)$ is a fuzzifying function defined over the crisp interval $[a, b]$, then for a given level of presumption α, the upper and lower bounds of the function can be represented by continuous curves. The integral of f over the interval $[a, b]$ is a fuzzy number defined by assigning the membership value α to the deterministic integrals of the upper bound curve $f_\alpha^+(x)$ and the lower bound curve $f_\alpha^-(x)$ for every level α. Thus, numerical integration of a fuzzy function over a crisp interval can be performed using conventional numerical

Table 1 Fuzzy representation of typical linguistic statement

Set	Linguistic statement	x 1	2	3	4	5	6
A	(Fiber content) low	0.0	0.2	0.4	0.6	0.8	1.0
A	Not low	1.0	0.8	0.6	0.4	0.2	0.0
A^2	Very low	0.0	0.04	0.16	0.36	0.64	1.0
A^4	Very very low	0.0	0.0016	0.0256	0.1296	0.4096	1.0
A^2	Not very low	1.0	0.96	0.84	0.64	0.36	0.0
B	(Fiber content) high	1.0	0.7	0.5	0.3	0.1	0.0
B^2	Very high	1.0	0.49	0.25	0.09	0.01	0.0
$A^2 \cup B^2$	Very low or very high	1.0	0.49	0.25	0.36	0.64	1.0
$A \cap \bar{A}^2$	Low but not very low	0.0	0.2	0.4	0.6	0.36	0.0

integration routines twice at each α-level. The procedure can be extended to evaluate the fuzzy integral (I) of a fuzzy function over a fuzzy interval. In this case, the limits of integration are fuzzy numbers. For a given α-level, there will be lower and upper bounds for the lower limit and upper limit of integration. Thus, there are four combinations of crisp intervals of integration. To find the α-cut bounds $[I^L, I^R]$, the procedure for the integration of a fuzzy function over a crisp interval outlined earlier is used to evaluate the integral for each of the four possible crisp intervals of integration. The lower limit I^L is simply the minimum of these four values resulting from the integration of $f_\alpha^-(x)$. The upper bound I^R is the maximum of these four values resulting from the integration of $f_\alpha^+(x)$. For example, if one is interested in evaluating the integral of a fuzzy function over a fuzzy interval at six α-levels, a total of 48 deterministic integral operations would have to be performed.

Systems of Fuzzy Equations

If fuzzy numbers are considered in terms of intervals of confidence at finite α-levels, the problem of solving systems of fuzzy equations can be reduced to solving systems of interval equations. The exact solution to a system of interval equations, that is, the interval vector with the smallest radius, is called the hull of the solution set. The computation of the hull of the solution set, in general, is extremely difficult. According to Neumaier,[16,17] for dimensions n larger than about 5 practical methods are available only in a special cases. The necessary and sufficient conditions for the existence of a unique solution to a single fuzzy algebraic equation are given by Zhao and Govind.[18,19] For the fuzzy expression $Ax = b$, the relative spread of b, defined as $(b^R/b^L)_\alpha$ for every level α, must be greater than or equal to the relative spread of A for a solution x to exist. The sufficient conditions state that for a system of fuzzy equations $Ax = b$, the degree of uncertainty in the matrix A determines the minimum degree of uncertainty that must be present in the vector b for a solution to exist. Thus, unlike the case for real valued equations, there is a dependence between the matrix A and the vector b.

Solution Procedure

If a solution exists for a system of fuzzy equations, it can also be found using the following optimization technique. For the system $Ax = b$, this method determines a fuzzy vector x that minimizes the differences in the bounds of Ax and b at finite α-levels. For a given α, the following unconstrained minimization problem can be formulated:

Design vector:

$$Y = \begin{Bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \\ y_{n+1} \\ \vdots \\ y_{2n} \end{Bmatrix} = \begin{Bmatrix} x_1^L \\ x_2^L \\ \vdots \\ x_n^L \\ \delta_1 \\ \vdots \\ \delta_n \end{Bmatrix} \tag{14}$$

where

$$y_i = x_i^L, \qquad y_{n+i}^2 = \delta_i^2 = x_i^R - x_i^L; \qquad i = 1, 2, \ldots, n$$

Objective function:
Minimize

$$f = \left(\Delta g_1^L\right)^2 + \left(\Delta g_1^R\right)^2 + \left(\Delta g_2^L\right)^2\left(\Delta g_2^R\right)^2 + \cdots + \left(\Delta g_n^L\right)^2 + \left(\Delta g_n^R\right)^2 \tag{15}$$

where

$$\left[g_1^L, g_1^R\right] = \left[a_{11}^L, a_{11}^R\right]\left[x_1^L, x_1^R\right] + \left[a_{12}^L, a_{12}^R\right] \times \left[x_2^L, x_2^R\right] + \cdots + \left[a_{1n}^L, a_{1n}^R\right]\left[x_n^L, x_n^R\right]$$

$$\left[g_2^L, g_2^R\right] = \left[a_{21}^L, a_{21}^R\right]\left[x_1^L, x_1^R\right] + \left[a_{22}^L, a_{22}^R\right] \times \left[x_2^L, x_2^R\right] + \cdots + \left[a_{2n}^L, a_{2n}^R\right]\left[x_n^L, x_n^R\right]$$

$$\vdots$$

$$\left[g_n^L, g_n^R\right] = \left[a_{n1}^L, a_{n1}^R\right]\left[x_1^L, x_1^R\right] + \left[a_{n2}^L, a_{n2}^R\right] \times \left[x_2^L, x_2^R\right] + \cdots + \left[a_{nn}^L, a_{nn}^R\right]\left[x_n^L, x_n^R\right]$$

$$\Delta g_1^L = g_1^L - b_1^L$$
$$\Delta g_1^R = g_1^R - b_1^R$$
$$\vdots$$
$$\Delta g_n^L = g_n^L - b_n^L$$
$$\Delta g_n^R = g_n^R - b_n^R$$

This problem can be solved using any of the standard unconstrained minimization methods.[20] Unconstrained minimization methods are iterative in nature. These algorithms start from an initial trial solution and proceed toward the minimum point in a sequential manner. The various algorithms differ primarily in the selection of search a directions. In this study, Powell's method was used. Powell's method is one of the most efficient zero-order methods that does not require the derivatives of the objective function and is based on the concept of conjugate directions.[20] Basically, this method first searches for the minimum of the function in n orthogonal directions, S_i, $i = 1, \ldots, n$ being the coordinate directions, where each search consists of updating the design vector

$$Y_{i+1} = Y_i + \lambda_i^* S_i$$

with Y_i denoting the starting vector, λ_i^* representing the scalar step size along the search direction S_i, and Y_{i+1} indicating the new design vector with an improved value of the objective function. Although these directions are not conjugate, they provide a starting point from which conjugate directions are generated. After completing n unidirectional searches, a new search direction is developed by connecting the first and last design vectors. This becomes the $n + 1$ search direction. In the next cycle, the first direction used in the previous cycle is discarded, and the search is continued along the remaining n directions. Then a new search direction is developed by connecting the first and last vectors of this cycle. This process is continued until the optimum point is found. Theoretically, when $n(n + 1)$ unidirectional searches are completed, the method would have generated n conjugate directions, and any quadratic function will be minimized by that time. This property, known as quadratic convergence, is achieved through the conjugate directions used in the procedure. However, for nonquadratic functions, the procedure might require more unidirectional searches. The unidirectional step lengths can be found using any of the one-dimensional optimization methods. The golden section method has been used in this work.[20] Several modifications have also been suggested to the basic Powell's method described to improve its convergence and reliability.

Table 2 Element data

Element no.	Area, in.^{2}	Length, in.	Modulus, psi
1	3.0	12.0	30.0×10^6
2	2.0	10.0	30.0×10^6
3	1.0	6.0	30.0×10^6

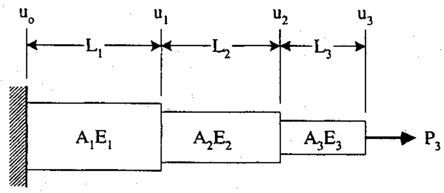

Fig. 4 Crisp description of stepped bar.

IV. Numerical Examples

A. Finite Element Analyses of a Bar

As a simple example of a fuzzy system, consider the bar shown in Fig. 4. The crisp system of equations, in this case, can be written as

$$\begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1+k_2 & -k_2 & 0 \\ 0 & -k_2 & k_2+k_3 & -k_3 \\ 0 & 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{Bmatrix} \tag{16}$$

where

$$k_j = \frac{A_j E_j}{L_j}; \qquad j = 1, 2, 3$$

u_i = displacement of node i
p_i = axial load applied to node $i\,(i = 0, 1, 2, 3)$, p_0 is the reaction force at the fixed end
L_j = length of element j
A_j = cross-sectional area of element
E_j = Young's modulus of element $j\,(j = 1, 2, 3)$

When the crisp boundary condition $u_0 = 0$ is applied, the system of equations reduces to

$$\begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} p_1 \\ p_2 \\ p_3 \end{Bmatrix} \tag{17}$$

The load vector and the corresponding displacement vector for the crisp case described in Table 2 are given by

$$P = \begin{Bmatrix} p_1 \\ p_2 \\ p_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 10{,}000 \end{Bmatrix} \text{lb}$$

$$U = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0.0013 \\ 0.0030 \\ 0.0050 \end{Bmatrix} \text{in.}$$

Three fuzzy situations are considered. In the first case (case A), only the load vector is treated fuzzy. For a solution to the fuzzy equations to exist, the zeros in the force vector must also be fuzzy. The second case (case B) is considered identical to the first with the exception that the modulus is also treated fuzzy. The third case (case C) is assumed to be identical to the second with the exception of fuzzy dimensions. The model parameters for each case are tabulated in the Appendix in the form of triangular fuzzy numbers. Two representative stiffness coefficients are depicted in Fig. 5. The

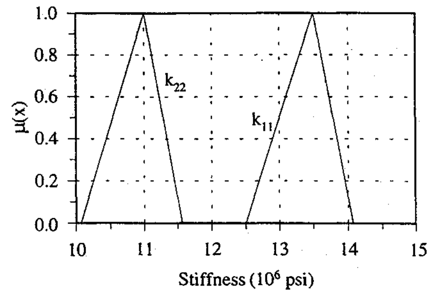

Fig. 5 Representative fuzzy stiffness coefficients for the bar.

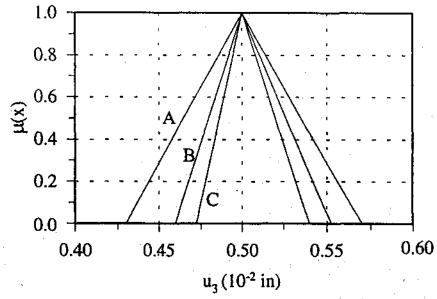

Fig. 6 Fuzzy displacement of the end of the bar.

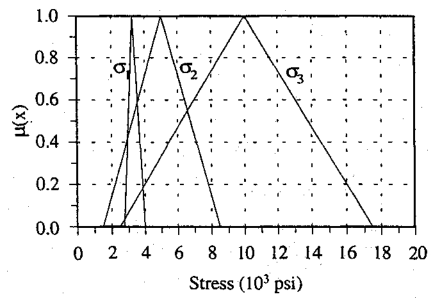

Fig. 7 Fuzzy stress in bar sections.

displacement u_3 obtained in each of these cases is shown in Fig. 6. The fuzzy stresses computed for each element in case C are shown in Fig. 7. Since the fuzziness in the load vector is the same in all three cases, the spread in the displacement can be observed to decrease as fuzziness is introduced in the stiffness matrix. The deterministic solution, that is, where $\mu = 1$, is the same for all three cases. This is because the modal values of the finite element model data were the same in each case.

B. Finite Element Analysis of a Beam

The governing equation for the deflection (w) of an Euler–Bernoulli beam is given by

$$\frac{d^2}{dx^2}\left(EI\frac{d^2 w}{dx^2}\right) = f(x) \qquad \text{for } 0 < x < L \tag{18}$$

The application of the variational principle leads to the element relations

$$K_{ij}^e = \int_{x_e}^{x_{e+1}} EI \frac{d^2 f_i^e}{dx^2}\frac{d^2 f_j^e}{dx^2} dx \tag{19}$$

$$F_i^e = \int_{x_e}^{x_{e+1}} f_i^e f\,dx + Q_i^e; \qquad i = 1, \ldots, 4; \qquad j = 1, \ldots, 4 \tag{20}$$

where f is the distributed force, Q_i^e are the generalized forces, and f_i^e are the interpolation functions given by[21]

$$f_1^e = 1 - 3\left(\frac{x - x_e}{h_e}\right)^2 + 2\left(\frac{x - x_e}{h_e}\right)^3 \tag{21}$$

$$f_2^e = -(x - x_e)\left(1 - \frac{x - x_e}{h_e}\right)^2 \tag{22}$$

$$f_3^e = 3\left(\frac{x - x_e}{h_e}\right)^2 - 2\left(\frac{x - x_e}{h_e}\right)^3 \tag{23}$$

$$f_4^e = -(x - x_e)\left[\left(\frac{x - x_e}{h_e}\right)^2 - \frac{x - x_e}{h_e}\right] \tag{24}$$

When the coordinates x_e, the element length h_e, the modulus E, the moment of inertia I, and the distributed load f are fuzzy, the fuzzy element stiffness matrix and force vector can be found using the fuzzy arithmetic and integration methods outlined. Using fuzzy addition, the element stiffness matrices and force vectors can be assembled to form the global system of equations in the usual manner. The resulting equation will be of the form

$$[K]\{u\} = \{F\} + \{Q\} \tag{25}$$

where $[K]$ is the global stiffness matrix, $\{u\}$ is the displacement vector, $\{F\}$ is the distributed force vector, and $\{Q\}$ is the generalized force vector.

Boundary Conditions

To investigate the concept of fuzzy boundary conditions, all supports are assumed to be elastic (modeled as linear springs) resisting both transverse and rotational deflections, and the spring constants are represented as fuzzy numbers. For a free condition, the spring constant is assigned a crisp zero value. To satisfy equilibrium at each node, the sum of the force generated by the spring and any applied point load must equal the internal force at the node. Thus, the components of the vector $\{Q\}$ will consist of these joint loads and the spring loads that are a function of $\{u\}$. To impose a particular condition on the global system of equations, the fuzzy spring constant is added to the diagonal term of the stiffness matrix that corresponds to the degree of freedom for which the boundary condition is applicable. The applied external point loads can be added directly to the force vector $\{F\}$. The resulting system of equations can be written as

$$[\tilde{K}]\{u\} = \{\tilde{F}\} \tag{26}$$

where $\tilde{K}$ is the revised stiffness matrix and $\tilde{F}$ is the force vector that includes the nodal point loads. This system of fuzzy equations can be solved as indicated earlier. Note that, for a solution to exist, the degree of fuzziness in the load vector $\tilde{F}$ must be sufficiently large to account for the uncertainty in the stiffness matrix.

Solution

Three cases of a fixed-fixed beam are considered (Fig. 8). In case A, the load is fuzzified to an overall support of ±10% of the crisp case. In case B, the same fuzzy load applied in case A is combined with a fuzzy modulus with a support of ±2% of the given crisp modulus. In case C, the fuzzy load is retained, the modulus is set back to the crisp value, and the boundary conditions are relaxed to "almost fixed." The model parameters for each case are listed in the form of triangular fuzzy numbers in the Appendix. Two elements were used in each case. Representative fuzzy load coefficients for case C are shown in Fig. 9. The maximum deflection in each of these cases is plotted in Fig. 10. As in the case of a bar, the fuzziness in the load vector is held constant in all cases. Thus, the spread in the displacement decreases as the fuzziness in the stiffness matrix increases. The increase in fuzziness of the stiffness matrix can be seen to result from the increase in uncertainty in Young's modulus (case B) or the uncertainty of the rigidity of the supports (case C). For both cases, this increase in uncertainty should result in an increase in the uncertainty of the displacement for the given load.

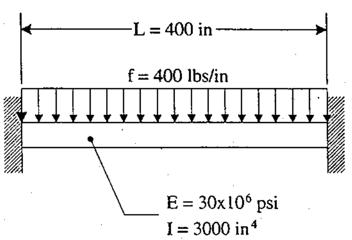

Fig. 8 Crisp description of beam.

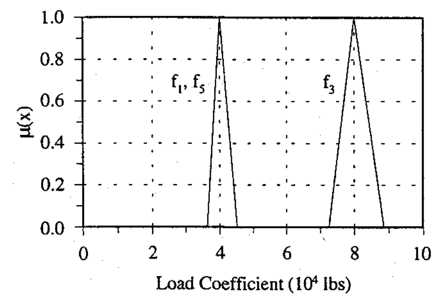

Fig. 9 Representative fuzzy load coefficients.

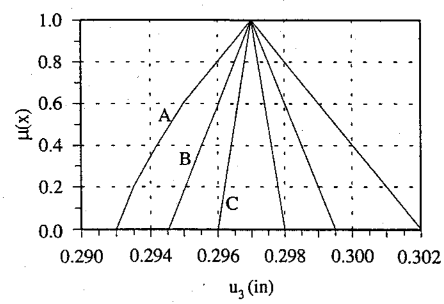

Fig. 10 Maximum deflection of fuzzy beam.

V. Conclusion

A methodology is presented for the finite element analysis of fuzzy systems. The feasibility of the method is demonstrated through simple bar and beam examples. The extension of the methodology for the solution of complex engineering analysis problems, including nonstructural applications, is currently under investigation. This methodology permits the use of a new philosophy for the solution of structural problems involving imprecisely defined geometry, external loads, and boundary conditions. By constructing membership functions for the imprecise quantities, the fuzzy calculus and integration techniques are used to derive the finite element equations. The resulting fuzzy system of equations are solved using the theory of interval equations. It is to be noted that fuzzy numbers are not linearly ordered and do not form a vector space. In a vector space, addition and scalar multiplication are required to satisfy the following rules[22]:

1) $x + y = y + x$
2) $x + (y + z) = (x + y) + z$
3) There is a unique zero vector such that $x + 0 = x$ for all x.
4) For each x there is a unique vector $-x$ such that $x + (-x) = 0$.
5) $1 \cdot x = x$
6) $(ab)x = a(bx)$
7) $a(x + y) = ax + ay$
8) $(a + b)x = ax + bx$

The set of all real numbers satisfies all of these properties, but fuzzy numbers do not. In particular, a fuzzy number in $\Re$ does not have an additive inverse, and the operation of fuzzy multiplication is, in general, not distributive.[18] Because of the lack of these properties, classical solution techniques, such as Gaussian elimination, cannot be extended to fuzzy equations. For larger finite element models, solution methods that are more robust and efficient will be required. Such methods are also being investigated.

Appendix Data for Bar and Beam Examples

Table A1 Data for bar examples as triangular fuzzy numbers

Parameters	Case A	Case B	Case C
A_1, in.^{2}	(3.00, 3.00, 3.00)	(3.00, 3.00, 3.00)	(2.99, 3.00, 3.01)
A_2, in.^{2}	(2.00, 2.00, 2.00)	(2.00, 2.00, 2.00)	(1.99, 2.00, 2.01)
A_3, in.^{2}	(1.00, 1.00, 1.00)	(1.00, 1.00, 1.00)	(0.99, 1.00, 1.01)
L_1, in.	(12.00, 12.00, 12.00)	(12.00, 12.00, 12.00)	(11.95, 12.00, 12.05)
L_2, in.	(10.00, 10.00, 10.00)	(10.00, 10.00, 10.00)	(9.95, 10.00, 10.05)
L_3, in.	(6.00, 6.00, 6.00)	(6.00, 6.00, 6.00)	(5.95, 6.00, 6.05)
E_1, psi	(3.0*e*7, 3.0*e*7, 3.0*e*7)	(2.8*e*7, 3.0*e*7, 3.1*e*7)	(2.8*e*7, 3.0*e*7, 3.1*e*7)
E_2, psi	(3.0*e*7, 3.0*e*7, 3.0*e*7)	(2.8*e*7, 3.0*e*7, 3.1*e*7)	(2.8*e*7, 3.0*e*7, 3.1*e*7)
E_3, psi	(3.0*e*7, 3.0*e*7, 3.0*e*7)	(2.8*e*7, 3.0*e*7, 3.1*e*7)	(2.8*e*7, 3.0*e*7, 3.1*e*7)
P_1, lb	(−9.0*e*3, 0.0, 9.0*e*3)	(−9.0*e*3, 0.0, 9.0*e*3)	(−9.0*e*3, 0.0, 9.0*e*3)
P_2, lb	(−1.5*e*4, 0.0, 1.5*e*4)	(−1.5*e*4, 0.0, 1.5*e*4)	(−1.5*e*4, 0.01, 1.5*e*4)
P_3, lb	(2.0*e*3, 1.0*e*4, 1.8*e*4)	(2.0*e*3, 1.0*e*4, 1.8*e*4)	(2.0*e*3, 1.0*e*4, 1.8*e*4)

Table A2 Data for beam examples as triangular fuzzy numbers

Parameter	Case A	Case B	Case C
L, in.	(400.0, 400.0, 400.0)	(400.0, 400.0, 400.0)	(400.0, 400.0, 400.0)
I, in.^{4}	(3.0*e*3, 3.0*e*3, 3.0*e*3)	(3.0*e*3, 3.0*e*3, 3.0*e*3)	(3.0*e*3, 3.0*e*3, 3.0*e*3)
E, psi	(3.0*e*7, 3.0*e*7, 3.0*e*7)	(2.94*e*7, 3.00*e*7, 3.06*e*7)	(3.0*e*7, 3.0*e*7, 3.0*e*7)
f, lb/in.	(360.0, 400.0, 440.0)	(360.0, 400.0, 440.0)	(360.0, 400.0, 440.0)

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